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LIMITS ON POPULATION GROWTH UNDER EXHAUSTIBLE RESOURCE CONSTRAINTS*

BY TAPAN MITRA[1]

1. INTRODUCTION

This paper is concerned with the following question: what patterns of population growth[2] are consistent with the attainment of some well-known social objectives, in the presence of exhaustible resource constraints?

In the literature on economic growth, with exhaustible resources as essential inputs in production, exogenous population is treated as either exponentially growing (see, for example, Solow [1974], Stiglitz [1974], Ingham and Simmons [1975]), or stationary (see Solow [1974], Stiglitz [1974], Dasgupta and Heal [1979]). Now, the first formulation (in the absence of technical progress) leads to problems right away, since per-capita consumption for *every* feasible program converges to zero (see Solow [1974], p. 40). Also, with a Classical Utilitarian objective, it leads to the non-existence of an optimal program (Ingham and Simmons [1975]). Thus the second formulation is found to be the more prevalent set-up.

Solow sums up this position as follows: "The convention of exponential population growth makes excellent sense as an approximation so long as population is well below its limit. On a time-scale appropriate to finite resources, however, exponential growth of population is an inappropriate idealization. But then we might as well treat the population as constant," (see Solow [1974], p. 36).

For reasons mentioned above, anyone who tries to formulate a growth model with exhaustible resources in an interesting way will fully appreciate this position. However, the last statement in the passage does seem to be somewhat abrupt, unless one believes that (say, for biological reasons) if population grows at all, it must do so exponentially!

It does seem to me to be worthwhile to explore, in a more systematic way, the exact *limits* on population growth under exhaustible resource constraints. When I am done, I hope to demonstrate, as a by-product, the special significance of an economy with zero population growth.

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[2] It should be noted that the problem is *not* to find an optimum population policy, treating population, or its growth rate, as a control variable. Rather, population is exogenously given, and the problem is to determine precisely what population profiles are consistent (or inconsistent) with an economy's welfare objectives. The problem of optimum population policies, with exhaustible resource constraints, is treated in papers by Koopmans [1974], Lane [1977], and Dasgupta and Mitra [1980].

In Sections 3 and 4, the paper examines the constraints that must be imposed on population growth, in order to attain the following two welfare objectives. One is the attainment of a non-trivial "maximin" program (for a definition, see Section 2). The other is the attainment of an "optimal" program in the Classical Utilitarian sense. More precisely, the paper obtains conditions on the production function, the utility function (in the "optimality" exercise) and the pattern of population growth which are *sufficient* for the existence of a non-trivial maximin program, and for the existence of an optimal program. Furthermore, parallel *necessary* conditions for the existence of non-trivial maximin and optimal programs are obtained, which are "close to" the sufficient conditions mentioned above. Thus, the results might be viewed as an "almost" complete answer to the question posed in the first paragraph of this paper.

As is to be expected, the necessary and sufficient conditions obtained for both welfare objectives, express the fact that population should not grow "too fast" (in a sense made precise in the statements of the theorems). The conditions are easily applicable to cases in which a pattern of population growth is parametrically specified, and this is demonstrated with some examples. These examples show that population growth in "quasi-arithmetic progression" (meaning that population at date t, $L_t=(t+1)^\lambda$ for $t\geq 0$, with $\lambda>0$) rather than "geometric progression" can be perfectly consistent with the attainment of our welfare objectives.

In Section 5, we examine two economies, which are identical in all respects except for their population growth patterns. If one economy always has at most as high a growth rate of population as the other, the first economy can enjoy at least as high a per-capita consumption level at each date as the second. As a result, we find that the first economy is at least as well-off as the second in terms of both the maximin and the optimality objective. This comparative dynamic exercise shows that, ceteris paribus, an economy with constant population (equal to the given initial population) is going to be at least as well-off as any other economy (with the same initial population) for which population is non-decreasing over time.

2. THE MODEL

Consider an economy with a technology given by a production function, G, from R_+^3 to R_+. The production possibilities consist of capital input, K, exhaustible resource input, D, labor input, L, and current output, $Z=G(K, D, L)$ for $(K, D, L)\geq 0$.[3]

Capital is non-depreciating, and the total output is $Y=G(K, D, L)+K$, for $(K, D, L)\geq 0$. *A total output function*, F, can then be defined by

$$F(K, D, L) = G(K, D, L) + K \qquad \text{for} \quad (K, D, L) \geq 0. \tag{2.1}$$

[3] For any two vectors, x and y, in R^N, $x\geq y$ means $x_i\geq y_i$ for $i=1,\ldots, N$; $x>y$ means $x\geq y$ and $x\neq y$; $x\gg y$ means $x_i>y_i$ for $i=1,\ldots, N$.

The production function, G, is assumed to satisfy:

(A.1) *G is concave, homogeneous of degree one, and continuously differentiable for* $(K, D, L) \gg 0$.

(A.2) $(G_K, G_D, G_L) \gg 0$ *for* $(K, D, L) \gg 0$.

The initial capital input, $\boldsymbol{K}$, and the initial available stock of the exhaustible resource, $\boldsymbol{S}$, are considered to be historically given, and positive. The available labor force (considered identical to "population"), $\boldsymbol{L}_t$, is exogenously given at each date, and satisfies

$$\boldsymbol{L}_0 = \boldsymbol{L},\ \boldsymbol{L}_{t+1} \geqq \boldsymbol{L}_t \quad \text{for} \quad t \geqq 0,\ \sup_{t\geq 0} [\boldsymbol{L}_{t+1}/\boldsymbol{L}_t] < \infty. \tag{2.2}$$

A *feasible program* is a sequence $\langle K, D, L, Y, C\rangle = \langle K_t, D_t, L_t, Y_{t+1}, C_{t+1}\rangle$ satisfying

$$\begin{aligned} &K_0 = \boldsymbol{K},\ \sum_{t=0}^{\infty} D_t \leqq \boldsymbol{S},\ L_t = \boldsymbol{L}_t && \text{for} \quad t \geq 0 \\ &Y_{t+1} = F(K_t, D_t, L_t),\ C_{t+1} = Y_{t+1} - K_{t+1} && \text{for} \quad t \geq 0 \\ &(K_t, D_t, L_t, Y_{t+1}, C_{t+1}) \geq 0 && \text{for} \quad t \geq 0. \end{aligned} \tag{2.3}$$

Associated with a feasible program $\langle K, D, L, Y, C\rangle$ is a *sequence of resource stocks* $\langle S\rangle = \langle S_t\rangle$, given by

$$S_0 = \boldsymbol{S},\ S_{t+1} = S_t - D_t \quad \text{for} \quad t \geq 0. \tag{2.4}$$

By (2.3), $S_t \geq 0$ and $S_{t+1} \leq S_t$ for $t \geq 0$. A feasible program $\langle K, D, L, Y, C\rangle$ is *interior* if $(K_t, D_t) \gg 0$ for $t \geq 0$. It is *regular interior* if $(K_t, D_t, C_{t+1}) \gg 0$ for $t \geq 0$.

Given a feasible program $\langle K, D, L, Y, C\rangle$, we denote

$$\begin{aligned} &k_t = (K_t/L_t),\ d_t = (D_t/L_t),\ g_{t+1} = (L_{t+1}/L_t) && \text{for} \quad t \geq 0 \\ &y_{t+1} = (Y_{t+1}/L_{t+1}),\ c_{t+1} = (C_{t+1}/L_{t+1}) && \text{for} \quad t \geq 0. \end{aligned} \tag{2.5}$$

A feasible program $\langle \bar{K}, \bar{D}, \bar{L}, \bar{Y}, \bar{C}\rangle$ *can maintain a positive per-capita consumption level* if

$$\inf_{t\geq 1} \bar{c}_t > 0. \tag{2.6}$$

It is *equitable* if

$$\bar{c}_t = \bar{c}_{t+1} \quad \text{for} \quad t \geq 1. \tag{2.7}$$

It is a *non-trivial equitable program* if it is equitable, and can maintain a positive per-capita consumption level. It is a *maximin program* if

$$\inf_{t\geq 1} \bar{c}_t \geq \inf_{t\geq 1} c_t \tag{2.8}$$

for every feasible program $\langle K, D, L, Y, C\rangle$. It is a *non-trivial maximin program* if it is a maximin program *and* can maintain a positive per-capita consumption level.

Felicity from consumption is obtained, according to a utility function, u, from R_+ to R. The utility function is assumed to have the following properties:

(A.3) *$u(c)$ is increasing in c for $c \geq 0$.*

(A.4) *$u(c)$ is concave and continuous for $c \geq 0$, and continuously differentiable for $c > 0$.*

(A.5) *$u'(c) \to \infty$ as $c \to 0$.*

A feasible program $\langle K^*, D^*, L^*, Y^*, C^* \rangle$ is *optimal* if

$$\limsup_{T \to \infty} \sum_{t=1}^{T} [L_t u(c_t) - L_t^* u(c_t^*)] \leq 0 \tag{2.9}$$

for every feasible program $\langle K, D, L, Y, C \rangle$.

3. THE MAXIMIN OBJECTIVE

The problem I will address in this section is to find necessary and sufficient conditions on the sequence $\langle \underset{\sim}{L}_t \rangle$ and the function G, such that there will exist a non-trivial maximin program.

A necessary condition for the existence of a non-trivial maximin program is the existence of a feasible program which can maintain a positive per-capita consumption level. Given our assumptions, it also turns out to be sufficient. Thus, the basic problem may be stated as follows: find necessary and sufficient conditions on the sequence $\langle L_t \rangle$ and function G, such that there will exist a feasible program which can maintain a positive per-capita consumption level.

The existence of a non-trivial maximin program also implies the existence of an efficient equitable program. Thus, once the basic problem (posed in the preceding paragraph) is solved, we will know the conditions under which the objectives of efficient allocation of resources and that of intergenerational equity do not conflict.

In order to obtain precise limitations on population growth, we have to specify the production function, G, in parametric form. Following Solow [1974], Stiglitz [1974] and others, I will assume that G is Cobb-Douglas:

$$G(K, D, L) = K^\alpha D^\beta L^\gamma, \ (\alpha, \beta, \gamma) \gg 0, \quad \alpha + \beta + \gamma = 1. \tag{A}$$

Given (A), (A.1) and (A.2) are clearly satisfied.

In a model of growth with exhaustible resources, the Cobb-Douglas or something "close to it" asymptotically, is often found to be the only really interesting case. For an elaboration of this point, see Solow [1974, p. 34].

My choice of (A) is, however, more for technical reasons. Even with (A), the answer to the problem I have posed is sufficiently involved, since we have imposed no parametric structure on the population path $\langle L_t \rangle$. An alternative route could be to place some parametric structure on $\langle \boldsymbol{L}_t \rangle$ (say, assume that $\boldsymbol{L}_t = (t+1)^\lambda$ with $\lambda \geq 0$), while keeping the production function, G, of a general form, given by (A.1) and (A.2).

My choice of the first route is a matter of taste. That is, I am more interested in finding the limitations on population growth, given a certain reasonable parametric technological structure, than in determining the technological requirements, given an ad hoc parametric population path structure.

Of course, the general problem is as I have posed it at the beginning of this section, with no parametric structure on either G or the sequence $\langle \boldsymbol{L}_t \rangle$. I doubt very much that the general problem can be solved as precisely as I have been able to solve the particular case. However, this is certainly an interesting open question.

Given (A), I will denote $[\gamma/(1-\beta)]$ by θ; $[\alpha/(1-\beta)]$ by δ; $[(1-\beta)/(1-\alpha)]$ by μ; $[(\alpha-\beta)/(1-\alpha)]$ by ν; $\sup_{t\geq 0} [\boldsymbol{L}_{t+1}/\boldsymbol{L}_t]$ by M; $\sum_{s=0}^{t} \boldsymbol{L}_s^\theta$ by A_t. For $0<e<\alpha$, denote $(\alpha-e)$ by a, $(\beta+e)$ by b; $[(1-b)/(1-a)]$ by m; $[(a-b)/(1-a)]$ by n.

Lemma 3.1. *Under (A), and $\alpha>\beta$, if e is a number such that $0<e<\alpha$, and $a>b$, then there is a feasible program $\langle K, D, L, Y, C\rangle$ and a scalar $E>0$ such that*

$$C_{t+1} \geq E(A_t^n/L_t^\delta) \quad \textit{for} \quad t \geq 0. \tag{3.1}$$

Proof. Choose $\phi>0$, with ϕ sufficiently close to zero to ensure that

$$e(1-a) + e(1-b) \geq \phi(1-a)b. \tag{3.2}$$

Clearly, this can be done. Then, it can be verified that

$$m\alpha - (1+\phi)\beta \geq n. \tag{3.3}$$

Denote $[\boldsymbol{L}_t^\theta / A_t^{(1+\phi)}]$ by H_t for $t\geq 0$. Then, by the Abel-Dini Theorem (see Knopp [1964, p. 299]), $\sum_{t=0}^{\infty} H_t < \infty$. Define $B^b = 2m(1+M)^n M^\theta$. Choose $\bar{h}>0$, such that

$$\sum_{t=0}^{\infty} \bar{h} B H_t = \boldsymbol{S}. \tag{3.4}$$

Define $h = \min\left[\frac{1}{2}, \bar{h}, \boldsymbol{K}\right]$.

Define a sequence $\langle K, D, L, Y, C\rangle$ as follows: $L_t = \boldsymbol{L}_t$ for $t\geq 0$; $K_0 = \boldsymbol{K}$, $K_t = hA_t^m$ for $t\geq 1$; $D_t = hBH_t$ for $t\geq 0$; $Y_{t+1} = G(K_t, D_t, L_t) + K_t$, $C_{t+1} = Y_{t+1} - K_{t+1}$ for $t\geq 0$. The sequence $\langle K, D, L, Y, C\rangle$ will be a feasible program, if we can show that $C_{t+1} \geq 0$ for $t\geq 0$.

For $t\geq 0$, $G(K_t, D_t, L_t) = h^\alpha A_t^{m\alpha} h^\beta B^\beta H^\beta L_t^\gamma = h^{(\alpha+\beta)} B^\beta A_t^{[m\alpha-(1+\phi)]} L_t^\theta \geq h^{(\alpha+\beta)} \cdot B^\beta A_t^n L_t^\theta$ [by (3.3)] $\geq hB^\beta A_t^n L_t^\theta$ (since $0<\alpha+\beta<1$ and $0<h<1$). Now, for $t\geq 0$, $(K_{t+1} - K_t) \leq hA_{t+1}^m - hA_t^m \leq hmA_{t+1}^n L_{t+1}^\theta$. For $t\geq 0$, we have

$$[A_{t+1}/A_t] = 1 + [L_{t+1}^\theta/A_t] \leq [1+M].$$

So $(K_{t+1}-K_t)\leq hm(1+M)^n A_t^n M^\theta L_t^\theta = \frac{1}{2} hB^\beta A_t^n L_t^\theta$. Therefore, $C_{t+1}\geq \frac{1}{2} hB^b A_t^n L_t^\theta$ for $t\geq 0$, and $\langle K, D, L, Y, C\rangle$ is a feasible program. Also, $(C_{t+1}/L_{t+1})\geq hB^b(A_t^n/L_t^\delta)(1/2M)$ for $t\geq 0$, so choosing $E=(hB^b/2M)$, (3.1) is satisfied. Q. E. D.

PROPOSITION 3.1[4]. *Under* (A), *there exists a feasible program which can maintain a positive per-capita consumption level if*

(3.5a) (i) $\alpha>\beta$

(3.5b) (ii) $\sup_{t\geq 0}[L_t^\delta/A_t^{\nu-\varepsilon}]<\infty$ *for some* $\varepsilon>0$.

PROOF. Choose $0<e<\alpha$, with e sufficiently close to zero, to ensure that $a>b$, and $n\geq \nu-\varepsilon$. By Lemma 3.1, there is a feasible program $\langle K, D, L, Y, C\rangle$ and a scalar $E>0$, such that (3.1) holds. Since $n\geq\nu-\varepsilon$, so by (3.5b), $\inf_{t\geq 0}(A_t^n/L_t^\delta)>0$, and so $\inf_{t\geq 0} c_{t+1}>0$. Q. E. D.

LEMMA 3.2. *Under* (*A*), *if* $\langle K, D, L, Y, C\rangle$ *is a feasible program*, *then there is a scalar* $0<V<\infty$, *such that*

$$K_t \leq VA_t^\mu \quad \text{for} \quad t\geq 0. \tag{3.6}$$

PROOF. Consider the pure accumulation program $\langle \hat{K}, \hat{D}, \hat{L}, \hat{Y}, \hat{C}\rangle$ given by $\hat{K}_0=K$, $\hat{K}_{t+1}=\hat{K}_t+G(\hat{K}_t, \hat{D}_t, \hat{L}_t)$ for $t\geq 0$; $\hat{L}_t=L_t$, $\hat{D}_t=D_t$ for $t\geq 0$; $\hat{Y}_{t+1}=\hat{K}_{t+1}$, $\hat{C}_{t+1}=0$ for $t\geq 0$. Then, for $t\geq 0$,

$$\hat{K}_{t+1}-\hat{K}_t=\hat{K}_t^\alpha \hat{D}_t^\beta \hat{L}_t^\delta. \tag{3.7}$$

Since $\hat{K}_{t+1}\geq\hat{K}_t$, so (3.7) yields

$$\hat{K}_{t+1}^{1-\alpha}-\hat{K}_t^{1-\alpha}\leq[\hat{K}_{t+1}-\hat{K}_t]/\hat{K}_t^\alpha=\hat{D}_t^\beta\hat{L}_t^\delta. \tag{3.8}$$

Writing $\hat{L}_t^\delta$ as $[\hat{L}_t^\theta]^{(1-\beta)}$, and using Holder's inequality in (3.8), we get for $T\geq 0$, $\hat{K}_{T+1}^{1-\alpha}-\hat{K}_0^{1-\alpha}\leq[\sum_{t=0}^T \hat{D}_t]^\beta[\sum_{t=0}^T \hat{L}_t^\theta]^{(1-\beta)}\leq S^\beta A_T^{(1-\beta)}$. Hence, $\hat{K}_{T+1}^{1-\alpha}\leq\hat{K}_0^{1-\alpha}+S^\beta A_T^{(1-\beta)}$ for $t\geq 0$. So, there is a number $0<V<\infty$, such that for $t\geq 0$,

$$\hat{K}_t\leq VA_t^{[(1-\beta)/(1-\alpha)]}=VA_t^\mu \tag{3.9}$$

Clearly, $K_t\leq\hat{K}_t$ for $t\geq 0$, so $K_t\leq VA_t^\mu$ for $t\geq 0$. Q. E. D

[4] It is possible to show that the following alternative set of conditions are also *sufficient* for the existence of a feasible program maintaining a positive per-capita consumption level:

(i) $\alpha>\beta$,

(ii) $\sum_{t=0}^\infty [L_t^{[(1-\gamma)/\beta]}/Q_t^{(\alpha/\beta)}]<\infty$

where $Q_t=\sum_{s=0}^t L_s$ for $t\geq 0$.

However, it is not known whether these conditions (or even something "close to" these conditions) are *necessary* for the existence of a feasible program maintaining a positive per-capita consumption level.

LEMMA 3.3. *Under (A), if there is an efficient program* $\langle K, D, L, Y, C\rangle$, *such that* $c_1>0$, *and* $c_{t+1}\geqq c_t$ *for* $t\geq 1$, *then*

(i) $c_{t+1}\leq k_t^\alpha d_t^\beta$ *for* $t\geq 0$, *and*

(ii) $\alpha>\beta$.

PROOF. By the proof of Proposition 3.2 of Dasgupta and Mitra [1979], $\langle K, D, L, Y, C\rangle$ is interior. Then, by Proposition 3.1 and Theorem 4.1 in Mitra [1978],

$$G_{D_{t+1}} = G_{D_t}F_{K_{t+1}} \quad \text{for} \quad t\geq 0. \tag{3.10}$$

I claim now that for $t\geq 0$,

$$c_{t+1} \leq k_t^\alpha d_t^\beta. \tag{3.11}$$

Suppose, on the contrary, there is some $t=\tau$ for which $c_{\tau+1}>k_\tau^\alpha d_\tau^\beta$, then $c_{\tau+1}=k_\tau^\alpha d_\tau^\beta+\varepsilon$, where $\varepsilon>0$. Note that, by feasibility,

$$k_{t+1} - k_t \leq g_{t+1}k_{t+1} - k_t = k_t^\alpha d_t^\beta - g_{t+1}c_{t+1}. \tag{3.12}$$

Now, $g_{\tau+1}c_{\tau+1}\geq c_{\tau+1}=k_\tau^\alpha d_\tau^\beta+\varepsilon$, so by (3.12), $k_{\tau+1}\leq k_\tau-\varepsilon<k_\tau$. Using (3.10), we have $G_{D_{t+1}}\geq G_{D_t}$, so that

$$(k_{t+1}^\alpha/k_t^\alpha) \geq (d_{t+1}^{1-\beta}/d_t^{1-\beta}) \quad \text{for} \quad t\geq 0. \tag{3.13}$$

Since $k_{\tau+1}<k_\tau$, so $d_{\tau+1}<d_\tau$ by (3.13). Hence, $k_{\tau+1}^\alpha d_{\tau+1}^\beta<k_\tau^\alpha d_\tau^\beta$, and so, $g_{\tau+2}c_{\tau+2}\geq c_{\tau+2}\geq c_{\tau+1}=k_\tau^\alpha d_\tau^\beta+\varepsilon>k_{\tau+1}^\alpha d_{\tau+1}^\beta+\varepsilon$. Hence, by (3.12), $k_{\tau+2}\leq k_{\tau+1}-\varepsilon$. Continuing this procedure for each succeeding period,

$$k_{t+1} \leq k_t - \varepsilon \quad \text{for} \quad t\geq \tau. \tag{3.14}$$

But (3.14) implies $k_t<0$ for large t, a contradiction. This establishes the claim made in (3.11), and proves (i).

By (i), we have for $t\geq 0$,

$$K_t^\alpha D_t^\beta L_t^\gamma \geq c_1 L_t. \tag{3.15}$$

By Lemma 3.2, we know that (3.6) holds. Using this in (3.15), we have, for $t\geq 0$,

$$D_t \geq [c_1{}^{(1/\beta)}L_t^{(1-\gamma)/\beta}]/[V^{(\alpha/\beta)}A_t^{(\alpha\mu/\beta)}]. \tag{3.16}$$

Now, $(1-\gamma)/\beta=(\alpha+\beta)/\beta>1>\gamma/(\alpha+\gamma)=\gamma/(1-\beta)=\theta$. And $[\alpha\mu/\beta]=[\alpha(1-\beta)/\beta(1-\alpha)]$. So by (3.16),

$$D_t \geq [c_1^{(1/\beta)}/V^{(\alpha/\beta)}]\,[L_t^\theta/A_t^{[\alpha(1-\beta)/\beta(1-\alpha)]}]. \tag{3.17}$$

Since $\sum_{t=0}^{\infty} D_t<\infty$, so by the Abel-Dini theorem (see Knopp [1964, p. 299]), $[\alpha(1-\beta)/\beta(1-\alpha)]>1$, that is $\alpha>\beta$, which proves (ii). Q.E.D.

LEMMA 3.4. *Under (A), and* $\alpha>\beta$, *if* $\langle K, D, L, Y, C\rangle$ *is an interior efficient program, then there is a scalar* $\hat{E}>0$, *such that*

(3.18) $$k_t^\alpha d_t^\beta \leq \hat{E}[A_t^\gamma/L_t^\delta] \quad \textit{for} \quad t \geq 0.$$

PROOF. Since $\langle K, D, L, Y, C\rangle$ is interior and efficient, so by Theorem 4.1 in Mitra [1978], $[K_t/G_{D_t}]\to 0$ as $t\to\infty$. That is, there is an integer $T<\infty$, such that for $t\geq T$, $[K_t/G_{D_t}]\leq 1$. Using this, we have for $t\geq T$

(3.19) $$[K_t/G_{D_t}] = [K_t^{1-\alpha}D_t^{1-\beta}/L_t^\gamma] \leq 1.$$

Now, (3.19) implies that, for $t\geq T$,

(3.20) $$D_t \leq [L_t^\theta/K_t^{(1/\mu)}].$$

Hence, for $t\geq T$, using Lemma 3.2, and (3.20) $G(K_t, D_t, L_t)=K_t^\alpha D_t^\beta L_t^\gamma \leq K_t^\alpha L_t^{(\beta\theta+\gamma)}/K_t^{(\beta/\mu)} = K_t^{[(\alpha-\beta)/(1-\beta)]}L_t^\theta \leq V^{[(\alpha-\beta)/(1-\beta)]}A_t^\gamma L_t^\theta$ (since $\alpha>\beta$). Thus, for $t\geq T$, we have,

(3.21) $$k_t^\alpha d_t^\beta \leq V^{[(\alpha-\beta)/(1-\beta)]}[A_t^\gamma/L_t^\delta].$$

Since $T<\infty$, it is possible to choose $\hat{E}>0$, such that (3.18) holds. Q.E.D.

PROPOSITION 3.2. *Under* (A), *if there is a feasible program which can maintain a positive per-capita consumption level, then*

(3.22a) (i) $\alpha > \beta$,

and

(3.22b) (ii) $\sup_{t\geq 0} [L_t^\delta/A_t^\gamma] < \infty.$

PROOF. If there exists a feasible program $\langle K^*, D^*, L^*, Y^*, C^*\rangle$ which can maintain a positive per-capita consumption level, then, clearly, there is an equitable program $\langle K', D', L', Y', C'\rangle$ with

$$[C'_t/L'_t] = \inf_{t\geq 1}[C_t^*/L_t^*] > 0 \qquad \text{for} \quad t \geq 1.$$

Then, by Theorem 1 of Dasgupta and Mitra [1979], there is an efficient equitable program $\langle K, D, L, Y, C\rangle$ with $c_t=c_1>0$ for $t\geq 1$.

By Lemma 3.3, (3.22a) holds. Also, (3.11) holds. Hence, by Lemma 3.4, (3.18) holds. Combining (3.11) and (3.18), we have for $t\geq 0$,

(3.23) $$c_1 \leq c_{t+1} \leq k_t^\alpha d_t^\beta \leq \hat{E}[A_t^\gamma/L_t^\delta].$$

It follows from (3.23) that (3.22b) holds. Q.E.D.

Remark. It should be noted that the method of proof actually ensures a stronger result than stated in Proposition 3.2; namely, if there exists a feasible program which can maintain a positive per-capita consumption level, then

(3.24a) (i) $\alpha > \beta$,

and

(3.24b) (ii) $\lim_{t\to\infty} [L_t^\delta/A_t^\gamma] = 0.$

The weaker form is retained in Proposition 3.2, merely for comparability with Proposition 3.1.

The importance of Propositions 3.1 and 3.2 in solving the problem addressed in this section is given in Theorem 3.1 and Corollary 3.1. The proofs are omitted, as a straightforward adaptation of the arguments used in Dasgupta and Mitra [1979, Theorem 1 and Corollary 1] is possible.

THEOREM 3.1. *Under (A), there exists and efficient equitable program if and only if there exists a feasible program which can maintain a positive per capita consumption level.*

COROLLARY 3.1. *Under (A), a feasible program is an efficient equitable program if and only if it is a non-trivial maximin program.*

Solow [1974] and Stiglitz [1974] have already shown (albeit in a continuous time model) that if (A) holds and $\boldsymbol{L}_t = \boldsymbol{L}n^t$, where $n \geq 1$, there exists a feasible program which can maintain a positive per-capita consumption level iff (i) $\alpha > \beta$, and (ii) $n = 1$. This result also follows directly from our Propositions 3.1 and 3.2.[5]

We now consider an example in which population grows to infinity over time, but still there exists a feasible program which can maintain a positive per capita consumption level.

EXAMPLE 3.1. Consider that (A) holds, $\alpha > \beta$, $\boldsymbol{L}_t = \boldsymbol{L}(t+1)^\lambda$, and $0 < \lambda < (\alpha/\beta) - 1$. (A numerical example would be the following: $G(K, D, L) = K^{0.20} D^{0.05} L^{0.75}$; so $\alpha = 0.20 > 0.05 = \beta$. Let $\boldsymbol{L}_t = \boldsymbol{L}(t+1)^2$ for $t \geq 0$; then $\lambda = 2$, so $0 < \lambda < (\alpha/\beta) - 1$ is satisfied, since $0 < 2 < (0.20/0.05) - 1 = 3$). Then, clearly $\boldsymbol{L}_t \to \infty$ as $t \to \infty$. Also, by Proposition 3.1, there exists a feasible program which can maintain a positive per-capita consumption level.[6] (Note that the numerical example we have given draws upon empirical evidence reported in Dasgupta and Heal [1979, p. 205 and p. 243].

4. THE UTILITARIAN OBJECTIVE

In this section, I will obtain necessary and sufficient conditions on the production function G, the utility function u, and the sequence $\langle \boldsymbol{L}_t \rangle$, such that there will exist an optimal program, in the Classical Utilitarian sense.

For this purpose, I will retain assumption (A) on G, and use the following parametric form of u:

[5] When $n = 1$, and G does not necessarily satisfy (A), (that is, G is a general neoclassical production function), the necessary and sufficient conditions for the existence of a feasible program maintaining a positive per-capita consumption level, are obtained by Cass and Mitra [1979].

[6] Consider that (A) holds, and $\boldsymbol{L}_t = \boldsymbol{L}(t+1)^\lambda$, where $\lambda > 0$. We note that by Proposition 3.2, if there exists a feasible program which can maintain a positive per-capita consumption level, then $\lambda \leq (\alpha/\beta) - 1$. Moreover, by using (3.24a) and (3.24b), one can get the more satisfactory result that "$\lambda < (\alpha/\beta) - 1$" itself is necessary and sufficient for the existence of a feasible program which can maintain a positive per-capita consumption level.

(B) $$u(c) = -(1/c^{\sigma}) \quad \text{where} \quad \sigma > 0.$$

Note that (B) clearly implies (A.3)–(A.5). Given (B), we will denote $-u(c)$ by $v(c)$.

Given (A) and (B), the conditions obtained provide us with precise limitations on population growth (in relation to the production and utility specifications) to be consistent with the attainment of the utilitarian objective.

PROPOSITION 4.1. *Under* (A), (B), *there exists an optimal program if*

(4.1a) (i) $\alpha > \beta$

(4.1b) (ii) $v\sigma > 1$

and

(4.1c) (iii) $\sum_{t=0}^{\infty} \frac{L_t^{(1+\sigma\delta)}}{A_t^{v\sigma-\hat{\varepsilon}}} < \infty$ *for some* $\hat{\varepsilon} > 0$

PROOF. By (4.1c), there is $\hat{\varepsilon}$, satisfying $v\sigma > \hat{\varepsilon} > 0$. Let ε denote $(\hat{\varepsilon}/\sigma)$; then $v > \varepsilon > 0$. Given (4.1a) choose $0 < e < \alpha$, with e sufficiently close to zero, to ensure that $a > b$, and $n \geq v - \varepsilon$.

By Lemma 3.1, there is a feasible program $\langle K, D, L, Y, C \rangle$, and a scalar $E > 0$, such that (3.1) holds. Since $n \geq v - \varepsilon$, so

(4.2) $$c_{t+1} \geq E(A_t^{v-\varepsilon}/L_t^{\delta}) \quad \text{for} \quad t \geq 0.$$

Then, for $t \geq 0$, $L_{t+1}v(c_{t+1}) = (L_{t+1}/L_t)L_t v(c_{t+1}) \leq ML_t[L_t^{\delta\sigma}/E^{\sigma}A_t^{\sigma v-\varepsilon\sigma}] \leq (M/E^{\sigma})\cdot [L_t^{(1+\delta\sigma)}/A_t^{\sigma v-\hat{\varepsilon}}]$.

Then, by (4.1c),

(4.3) $$\sum_{t=0}^{\infty} L_{t+1}v(c_{t+1}) < \infty.$$

Then, by Brock and Gale [1969, Lemma 2], there exists an optimal program. Q.E.D.

PROPOSITION 4.2. *Under* (A), (B), *if there exists an optimal program, then*

(4.4a) (i) $\alpha > \beta$

(4.4b) (ii) $v\sigma > 1$

and

(4.4c) (iii) $\sum_{t=0}^{\infty} \frac{L_t^{(1+\sigma\delta)}}{A_t^{v\sigma}} < \infty.$

PROOF. Suppose there is an optimal program, call it $\langle K, D, L, Y, C \rangle$. Then, for each $t \geq 1$, the expression $L_t u\{[F(K_{t-1}, D_{t-1}, L_{t-1}) - K]/L_t\} + L_{t+1}u\{[F(K, D_t, L_t) - K_{t+1}]/L_{t+1}\}$ must be a maximum at $K = K_t$. By (A), (B), the maximum must be at an interior point; that is $(C_{t+1}, K_t, D_t) \gg 0$ for $t \geq 0$. So, (4.5) $$u'(c_t) = u'(c_{t+1})F_{K_t} \quad \text{for} \quad t \geq 1.$$

Also, for each $t \geq 1$, and for $0 < D < D_{t-1} + D_t$, the expression $L_t u\{[F(K_{t-1}, D, L_{t-1}) - K_t]/L_t\} + L_{t+1} u\{[F(K_t, D_{t-1} + D_t - D, L_t) - K_{t+1}]/L_{t+1}\}$ must be a maximum at $D = D_{t-1}$. Since the maximum is at an interior point, so

(4.6) $$u'(c_t)F_{D_{t-1}} = u'(c_{t+1})F_{D_t} \quad \text{for} \quad t \geq 1.$$

Define a sequence $\langle p, q \rangle$ in the following way:

(4.7) $$\begin{cases} p_0 = u'(c_1)F_{k_0};\ p_t = u'(c_t) & \text{for} \quad t \geq 1 \\ q_t = u'(c_1)F_{D_0} & \text{for} \quad t \geq 0. \end{cases}$$

Note that by (4.6), (4.7),

(4.8) $$p_{t+1} = [q_0/F_{D_t}] \quad \text{for} \quad t \geq 0.$$

By (4.5), $u'(c_t) \geq u'(c_{t+1})$, so $c_{t+1} \geq c_t$ for $t \geq 0$. Hence, by Lemma 3.3 [using the fact that $\langle K, D, L, Y, C \rangle$ is efficient], (4.4a) holds, and

(4.9) $$c_{t+1} \leq k_t^{\alpha} d_t^{\beta} \quad \text{for} \quad t \geq 0.$$

Using (4.8) and Corollary 4.1 in Mitra [1978],

(4.10) $$\sum_{t=0}^{\infty} p_{t+1} C_{t+1} < \infty.$$

Using (4.7) in (4.10),

(4.11) $$\sum_{t=1}^{\infty} L_t u'(c_t) c_t < \infty.$$

Using (B), $u'(c)c = \sigma c^{-\sigma} = \sigma v(c)$ for $c > 0$.
Hence, by (4.11), we have (since $L_{t+1} \geq L_t$ for $t \geq 0$)

(4.12) $$\sum_{t=1}^{\infty} L_t v(c_{t+1}) < \infty.$$

By (4.9), and (3.18) [using Lemma 3.4],

(4.13) $$c_{t+1} \leq \hat{E}[A_t^{\nu}/L_t^{\delta}] \quad \text{for} \quad t \geq 0.$$

Using (4.13) in (4.12), we have

(4.14) $$\sum_{t=1}^{\infty} \frac{L_t^{(1+\delta\sigma)}}{\hat{E}^{\sigma} A_t^{\nu\sigma}} < \infty.$$

Now, (4.4c) follows directly from (4.14). Since $(1+\sigma\delta) > 1 > \theta$, so (4.4c) implies by the Abel-Dini Theorem, that $\nu\sigma > 1$, which establishes (4.4b). Q. E. D.

Remark.[7] Note that if an optimal program exists, then the period social wel-

[7] Given the technique of analysis in Sections 3 and 4, it should be fairly clear how technical

(*Continued on next page*)

fares $[L_t u(c_t)]$ are summable, by (4.12). We will use this fact later, in Section 5.

Solow [1974] and Dasgupta and Heal [1979] have shown (in a continuous time model) that if (A) and (B) hold, and $L_t = Ln^t$, where $n \geq 1$, there exists an optimal program iff (i) $\alpha > \beta$, (ii) $n = 1$, and (iii) $\nu\sigma > 1$. This result also follows directly from our Propositions 4.1 and 4.2.

We now consider an example in which population grows to infinity over time, and still there exists an optimal program.

EXAMPLE 4.1. Consider that (A), (B) hold, $\alpha > \beta$, $\nu\sigma > 1$, $L_t = L(t+1))^\lambda$, and $0 < \lambda < [\sigma\alpha/(1-\alpha+\beta\sigma)] - 1$. {A numerical example would be the following: $G(K, D, L) = K^{0.20} D^{0.05} L^{0.75}$; $u(c) = -(1/c^{24})$; $L_t = L(t+1)$. Then $\alpha = 0.20 > 0.05 = \beta$; $\sigma = 24$, $\nu = (3/16)$, so $\nu\sigma = (9/2) > 1$; $[\sigma\alpha/(1-\alpha+\beta\sigma)] = (4.8/2) = 2.4$; $\lambda = 1$. Since $0 < 1 < 2.4 - 1$, so λ satisfies the inequality $0 < \lambda < [\sigma\alpha/(1-\alpha+\beta\sigma)] - 1$.} Then, it follows from Proposition 4.1, that there exists an optimal program.[8] Note also that $L_t \to \infty$ as $t \to \infty$, since $\lambda > 0$.

5. A COMPARATIVE DYNAMIC EXERCISE

Consider two economies, with the same production function specified by (A), and the same utility function specified by (B). The two economies have the same initial stocks of capital, labor and exhaustible resource, $\boldsymbol{K}$, $\boldsymbol{L}$, $\boldsymbol{S}$. The population paths of both economies satisfy the restrictions given by (2.2). The difference between the two economies is that the growth rate of population in the first is always at most as high as that in the second. Can we then say, in some precise sense, that the first economy is at least as well off as the second? This is the subject matter of this section.

Let us denote the variables of the first economy with a superscript of 1, those of the second with a superscript of 2. Assume that

$$g^1_{t+1} \leq g^2_{t+1} \quad \text{for} \quad t \geq 0. \tag{5.1}$$

We then have the following result.

PROPOSITION 5.1. *Under (A), if (5.1) holds, and $\langle K^2, D^2, L^2, Y^2, C^2 \rangle$ is any feasible program for the second economy, there is a feasible program $\langle K^1, D^1,$*

(*Continued*)
progress (not necessarily exponential) can be handled in this framework. More precisely, our analysis could be used to answer a question of the following sort: "What patterns of population growth *and* technical progress are consistent with the attainment of some well-known social objectives, in the presence of exhaustible resource constraints?" When both population growth and technical progress are exponential, such a question has been answered by Stiglitz [1974], and Mitra [1981].

[8] Suppose (A), (B) hold and $L_t = L(t+1)^\lambda$, where $\lambda > 0$. It follows from Proposition 4.2, that if there exists an optimal program, then (i) $\alpha > \beta$, (ii) $\nu\sigma > 1$, and (iii) $\lambda < [\sigma\alpha/(1-\alpha+\beta\sigma)] - 1$. This, together with our conclusion in Example 4.1, shows that conditions (i), (ii) and (iii) are necessary and sufficient conditions for the existence of an optimal program.

L^1, Y^1, $C^1\rangle$ *for the first economy, with*

$$c^1_{t+1} \geq c^2_{t+1} \quad \text{for} \quad t \geq 0. \tag{5.2}$$

PROOF. Define a sequence $\langle K^1, D^1, L^1, Y^1, C^1\rangle$ as follows. Let $L^1_t = \boldsymbol{L}^1_t$ for $t \geq 0$; $K^1_t = k^2_t L^1_t$, $D^1_t = D^2_t$, $Y^1_{t+1} = F(K^1_t, D^1_t, L^1_t)$, $C^1_{t+1} = Y^1_{t+1} - K^1_{t+1}$ for $t \geq 0$. Then, for $t \geq 0$,

$$\begin{aligned} C^1_{t+1} &= G(K^1_t, D^1_t, L^1_t) + K^1_t - K^1_{t+1}, \quad \text{or} \\ c^1_{t+1} &= \left(\frac{L^1_t}{L^1_{t+1}}\right) [G(k^1_t, d^1_t, 1) + k^1_t] - k^1_{t+1} \\ &\geq \left\{\frac{[G(k^2_t, d^2_t, 1) + k^2_t]}{g^1_{t+1}}\right\} - k^2_{t+1} \\ &\geq \left\{\frac{[G(k^2_t, d^2_t, 1) + k^2_t]}{g^2_{t+1}}\right\} - k^2_{t+1} \\ &= c^2_{t+1} \geq 0. \end{aligned}$$

Hence, $\langle K^1, D^1, L^1, Y^1, C^1\rangle$ is a feasible program for the first economy. Furthermore, (5.2) is clearly satisfied. Q.E.D.

Suppose, now, that for each economy a non-trivial maximin program and an optimal program exist. We denote by W^1_M the maximin per-capita consumption level, and by W^1_U the sum of per-period social welfares on the optimal program, for the first economy. The corresponding magnitudes of the second economy are W^2_M and W^2_U.

PROPOSITION 5.2. *Under* (*A*), (*B*), *if* (5.1) *holds, then*

(i) $W^1_M \geq W^2_M$,

and

(ii) $W^1_U \geq W^2_U$.

PROOF. That (i) is true follows trivially from Proposition 5.1. To prove (ii), note that $L^2_t \geq L^1_t$ for $t \geq 0$, by (5.1). If $\langle \bar{K}^2, \bar{D}^2, \bar{L}^2, \bar{Y}^2, \bar{C}^2\rangle$ is an optimal program for the second economy, then there is a feasible program $\langle K^1, D^1, L^1, Y^1, C^1\rangle$ for the first economy, such that $u(c^1_t) \geq u(\bar{c}^2_t)$ for $t \geq 1$ (by Proposition 5.1). Since $u(c) < 0$ for $c \geq 0$, so $L^1_t u(c^1_t) \geq L^1_t u(\bar{c}^2_t) \geq \bar{L}^2_t u(\bar{c}^2_t)$ for $t \geq 0$. Consequently, if $\langle \bar{K}^1, \bar{D}^1, \bar{L}^1, \bar{Y}^1, \bar{C}^1\rangle$ is an optimal program for the first economy,

$$\sum_{t=1}^{\infty} \bar{L}^1_t u(\bar{c}^1_t) \geq \sum_{t=1}^{\infty} \bar{L}^2_t u(\bar{c}^2_t).$$

This establishes (ii). Q.E.D.

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